KEK Preprint 99-160
January 2000
hep-th/0001011

Noncommutativities of D-branes and θ-changing Degrees of Freedom in D-brane Matrix Models

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Abstract

It is known that when there are several D-branes, their space-time coordinates in general become noncommutative. From the point of view of noncommutative geometry, it reflects noncommutativity of the world volume of the D-branes. On the other hand, as we showed in the previous work, in the presence of the constant antisymmetric tensor field the momentum operators of the D-branes have noncommutative structure. In the present paper, we investigate a relation between these noncommutativities and the description of D-branes in terms of the noncommutative Yang-Mills theory recently proposed by Seiberg and Witten. It is shown that the noncommutativity of the Yang-Mills theory, which implies that of the world volume coordinates, originates from both noncommutativities of the transverse coordinates and momenta from the viewpoint of the lower-dimensional D-branes. Moreover, we show that this noncommutativity is transformed by coordinate transformations on the world volume and thereby can be chosen in an arbitrary fixed value. We also make a brief comment on a relation between this fact and a hidden symmetry of the IIB matrix models.

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1 Introduction

Candidates for the nonperturbative definition of string theory have been proposed for these few years[1, 2, 3]. Although these matrix models have passed several nontrivial checks so far, they are far from satisfactory because not only they have little predictions as to nonperturbative effects of string theory, but they do not completely reproduce the perturbative string theory. These failure might result from the lack of information on fundamental degrees of freedom and symmetry or principle which governs them in these models. Since matrix models start with the lower-dimensional D-branes such as D-instantons or D-particles as fundamental degrees of freedom (although there is a difference in their interpretations), it is important to examine the physics of D-branes and survey true fundamental degrees of freedom and a hidden symmetry in the nonperturbative string theory.

One of the most appealing features of D-branes is their noncommutative structure. Namely, when there are several D-branes which are parallel to each other, their transverse coordinates are promoted to non-commuting matrices[4]. From the point of view of noncommutative geometry, this noncommutativity can be considered to reflect the noncommutative structure of the world volume of D-branes, since the transverse coordinates are functions on the world volume of D-branes which is naturally described by them. In what follows, we will refer to this noncommutativity as *transverse noncommutativity*.

On the other hand, we found in our previous work[5] that in the presence of the background antisymmetric tensor field along the transverse directions the momentum operators of the D-brane toward these directions become noncommutative.[†] Henceforth this noncommutativity will be referred to as *momentum noncommutativity*. These two noncommutativities are dual to each other in the sense that the former realizes the noncommutative structure of the transverse coordinates of world volume, while the latter realizes that of the conjugate momenta. Moreover, as shown in [5], when the transverse directions are compactified on the two-torus they satisfy non-trivial relations. These results are in nonperturbative aspects of string theory and should be naturally considered to reflect a hidden symmetry or an unknown mechanism of string theory. In fact, from the viewpoint of the underlying principle of string theory, there exist a proposal that regardless of fundamental strings or D-branes, their transverse directions and longitudinal ones have uncertainty of order of string scale and that play dual roles to each other, which is a manifestation of the fundamental principle or symmetry of string theory[8, 9, 10]. It is natural to guess that noncommutativities of D-branes are closely related to this idea. Therefore further studies of these noncommutativities would shed light on future investigations.

Recently Seiberg and Witten analyzed the open string theory in the presence of the constant Neveu-Schwarz 2-form field B_{ij} and showed that its low-energy effective theory is described by the ordinary Yang-Mills (YM) theory when one adopts the Pauli-Villars regularization, while it is described by the noncommutative Yang-Mills (NCYM) theory with a parameter of noncommutativity given by $\theta = B^{-1}$ in the case of point-splitting regularization and thus both are physically equivalent [11]. Here the NCYM theory with

[†] Accordingly Matrix theory or a low energy effective field theory of D-branes on a torus in the constant antisymmetric field background is described by a gauge theory on a noncommutative torus[6, 7].

θ is defined by replacing all the ordinary product of functions such as gauge fields with what is called the $*$-product

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{ij}\frac{\partial}{\partial \xi^i}\frac{\partial}{\partial \zeta^j}} f(x+\xi)g(x+\zeta)|_{\xi=\zeta=0}, \tag{1.1}$$

where θ^{ij} is a real constant number. Noncommutative gauge symmetry with gauge parameter $\hat{\lambda}$ for the noncommutative gauge field $\hat{A}_i$ is given as

$$\hat{\delta}_{\hat{\lambda}}\hat{A}_i = \partial_i\hat{\lambda} + i\hat{\lambda} * \hat{A}_i - i\hat{A}_i * \hat{\lambda}. \tag{1.2}$$

These implies the following non-trivial commutation relation of the base space coordinate x^i:

$$[x^i, x^j] = i\theta^{ij}. \tag{1.3}$$

In [11], based on these observations, a concrete map from ordinary gauge fields to noncommutative ones is constructed. Furthermore, it is conjectured that there exists a series of equivalent NCYM theories with arbitrary values of θ and an interesting prediction is made that such a NCYM theory with θ between $\theta = 0$ (ordinary) and $\theta = B^{-1}$ corresponds to a suitable regularization which interpolates between Pauli-Villars and point-splitting. In the following, we will call the noncommutativity θ of NCYM theory *longitudinal noncommutativity* because it is caused by the noncommutativity of the coordinates of the base space as in (1.3) which are nothing but the longitudinal coordinates of the D-brane world volume as shown explicitly in [11, 12, 13, 14].

For an attempt to define string theory nonperturbatively from lower-dimensional D-branes as a fundamental constituent like [2, 9, 15, 16], it would be meaningful to reinterpret the above results in terms of them. In fact, it is shown that higher-dimensional D-branes can be regarded as a configuration of infinitely many lower-dimensional ones with the transverse noncommutativity [17, 18, 19, 20]. In section 2, a D-string which is made from infinitely many D-instantons in the presence of a constant B_{ij} background are considered as a simple example, which has been analyzed in our previous paper[5]. Then We apply the result in [11] to this configuration and interpret it in terms of D-instanton degrees of freedom. In particular, we clarify the interrelation between the above-mentioned noncommutativities. Similar analysis was also done in a recent paper[21]. In section 3, we focus on the degrees of freedom to change the value of θ suggested in [11] and refine it from the standpoint of the longitudinal noncommutativity. Section 4 is devoted to discussions on the meaning of our results for a hidden symmetry of matrix models.

2 Interrelation between Noncommutativities

In this section we consider a configuration of infinitely many D-instantons in flat space with a metric G'_{ij} in the presence of a constant 2-form field B'_{ij} background. For simplicity, the coordinates of D-instanton q^i ($\infty \times \infty$ matrices) are assumed to satisfy

$$[q^1, q^2] = -ik, \tag{2.1}$$

where k is a real number. In this case, since the components of B'_{ij} not along the 1,2-directions can be gauge away, we can assume that $B'_{12} = -B'_{21} \neq 0$ and B'_{ij}=0 otherwise.

From now on indices i, j are understood to be 1 or 2. k is nothing but a parameter which measures the transverse noncommutativity of D-instantons. On the other hand, B'_{ij} parametrizes the momentum noncommutativity since the momentum operator of D-instantons π_i satisfies

$$[\pi_1, \pi_2] = iB'_{12}, \tag{2.2}$$

when $k = 0$[5].[‡] When $k \neq 0$, (2.2) are modified as

$$[\pi_1, \pi_2] = i\frac{B'_{12}}{1 - kB'_{12}}. \tag{2.3}$$

However, even if $k = 0$, π_i still has the noncommutativity by the presence of B'_{12}. Therefore we also regard it as a parameter of the momentum noncommutativity in the case of non-zero k.

As shown in [5, 21], as a background of string, our configuration of D-instantons is equivalent to a D-string in the background g_{ij}, B_{ij}, F_{ij} given by[§]

$$g_{ij} = G'_{ij}, \qquad B_{ij} + F_{ij} = B'_{ij} + \frac{1}{k}\epsilon_{ij}.^{\P} \tag{2.4}$$

In these equations, in the D-instanton picture in the right hand side a noncommutative world volume spanned by q^1, q^2 is constructed, while in the left hand side a D-string world volume in the ordinary static gauge is considered. In fact, if we set

$$B_{ij} = B'_{ij}, \quad F_{ij} = \frac{1}{k}\epsilon_{ij}, \tag{2.5}$$

by doing the appropriate gauge fixing, we find that the D-instanton picture corresponds to the choice of the coordinate (parametrization) on the D-string world volume in which $U(1)$ gauge field strength F_{ij} is always equal to ϵ_{ij}/k [17, 18, 22]. As we will see in the next section, if we choose a world volume coordinate using the general coordinate transformation on the world volume in such a way that F_{ij} is a constant, it turns out to have noncommutative structure in general. Therefore, since the world volume in the D-instanton picture is noncommutative, we should apply the result in [11] to the left hand side of (2.4), namely, a D-string in the static gauge.

Seiberg and Witten began with the open string theory in the background of a D-brane and B_{ij} along its world volume described by the action

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma g_{ij}\partial_a x^i \partial^a x^j - \frac{i}{2}\int_{\partial\Sigma} B_{ij} x^i \partial_t x^j, \tag{2.6}$$

[‡] In [5] D-instantons are compactified on the T^2 with the B_{ij} flux and (2.2) is obtained reflecting the global structure of the torus. Thus it is likely that the existence of components of B'_{ij} along compact directions which cannot be gauged away is essential to define the momentum noncommutativity in a well-defined manner.

[§] Although in [5] the configuration is compactified on T^2 as mentioned in the above footnote, a similar argument also works in this case.

[¶] In order to match the notation with those in [11], we change the normalizations of B'_{ij}, B_{ij} from those in [5] by a factor $2\pi\alpha'$.

and showed that in the point-splitting regularization on the world sheet the low-energy effective theory is given by the NCYM theory with metric G_{ij} and noncommutativity parameter θ^{ij} given as follows:

$$\frac{1}{G} = -\frac{\theta}{2\pi\alpha'} + \frac{1}{g + 2\pi\alpha' B}. \tag{2.7}$$

Moreover, from the fact that in the Pauli-Villars regularization it is described by the ordinary YM theory ($\theta = 0$), they pointed out that both YM theories arise from the same two-dimensional field theory regularized in different ways and are physically equivalent. Putting this observation forward, they predicted that for all values of θ there exists an equivalent description of NCYM theory. This degrees of freedom is conjectured to be captured by introducing a two-form field Φ as

$$\frac{1}{G + 2\pi\alpha'\Phi} = -\frac{\theta}{2\pi\alpha'} + \frac{1}{g + 2\pi\alpha' B}. \tag{2.8}$$

Although setting $\Phi = 0$ reproduces (2.7), in the general case of $\Phi \neq 0$ we get NCYM theory with a different value of θ. Φ corresponds to the degrees of freedom of a magnetic background in the NCYM theory and its existence is naturally required from the Morita equivalence which is an equivalent relation between different NCYM theories[6, 23, 24]. In [24] the degrees of freedom to change Φ and θ in a suitable way keeping the NCYM theory physically invariant are argued to exist. From the point of view of the noncommutativities of D-branes, as mentioned in the introduction, the noncommutativity θ of the NCYM theory reflects that of the base space coordinate. In the present example the base space is nothing but the D-string world volume and in this sense θ can be considered to parametrize the longitudinal noncommutativity.

On our D-string world volume there is a constant $U(1)$ gauge field strength as in (2.5) and thereby the second term in (2.6) are modified as $B'_{ij} \to B'_{ij} + \epsilon_{ij}/k$. Noting this, we obtain from (2.4), (2.8)

$$\frac{1}{G' + 2\pi\alpha'(B' + \epsilon/k)} = \frac{1}{G + 2\pi\alpha'\Phi} + \frac{\theta}{2\pi\alpha'}. \tag{2.9}$$

This is the equation which relates the moduli of the D-instanton configuration G', B', k to the parameters in the NCYM theory G, θ, Φ. A similar formula was also obtained in [21]. In what follows, we will define $\mathcal{F}'$ as

$$\mathcal{F}' \equiv B' + \frac{1}{k}\epsilon. \tag{2.10}$$

In some particular cases, θ and Φ are expressed in a simple form from (2.9):

(1) $\Phi = 0$.

$$\frac{1}{\theta^{12}} = -\mathcal{F}'_{12} - \frac{\det G'}{(2\pi\alpha')^2 \mathcal{F}'_{12}}, \tag{2.11}$$

this corresponds to the NCYM theory obtained in the point-splitting regularization in [11]

(2) $\theta = 0$.

$$\Phi = \mathcal{F}', \tag{2.12}$$

this corresponds to the ordinary YM theory (however, with the magnetic background Φ) obtained in the Pauli-Villars regularization.
(3) $\theta = 1/\mathcal{F}'$.

$$\Phi = -\mathcal{F}', \tag{2.13}$$

this corresponds to the exact solution or the solution in the zero-slope limit argued in [11]

$$\alpha' \sim \epsilon^{\frac{1}{2}} \to 0, \qquad g_{ij} \sim \epsilon \to 0. \tag{2.14}$$

In fact, in [11] one of the exact solutions to (2.8) is proposed as

$$\theta = \frac{1}{B}, \quad G = -(2\pi\alpha')^2 B \frac{1}{g} B, \quad \Phi = -B, \tag{2.15}$$

and it is pointed out that this is also the solution in the limit (2.14).

In the intermediate region (general case) of interest at present, (2.9) leads to a quadratic equation for Φ_{12} and its solution is determined in a unique way from the requirement that it should reproduce the above special cases. It reads

$$\Phi_{12} = \frac{1}{\theta^{12}} - \frac{\mathcal{F}'_{12} + 1/\theta^{12}}{(\mathcal{F}'_{12}\theta^{12} + 1)^2 + \det G' \left(\theta^{12}/2\pi\alpha'\right)^2}. \tag{2.16}$$

This is the most general exact solution to (2.9) in our case. For given $\mathcal{F}'$, if we choose θ in an arbitrary fixed value, we can determine Φ according to this equation. In the low-energy limit (zero-slope limit), as we see above, $\theta^{12} = -1/(B'_{12} + 1/k)$ and in this sense the transverse noncommutativity k and the momentum noncommutativity B'_{12} of the D-instantons are encoded into the longitudinal noncommutativity θ of the D-string in a special combination. However, in general cases, the Φ degrees of freedom make their relation complicated as in (2.16). Note that in such cases the situation that transverse and momentum noncommutativity are combined into the longitudinal one only in the special combination is the same. This fact seems to imply that in defining the nonperturbative string theory in terms of the D-instanton degrees of freedom, it is necessary to introduce not only the transverse noncommutativity as has been formulated so far, but also the momentum noncommutativity on an equal footing.

3 The General Coordinate Transformation on the D-string World Volume and the Φ Degrees of Freedom

In this section from the point of view of the D-string world volume theory we reconsider the degrees of freedom of Φ conjectured in [11] which changes the value of θ keeping NCYM theory intact. As discussed in [17, 18, 22], it is known that if we choose the coordinate on the world volume in such a way that the $U(1)$ field strength $\mathcal{F}_{ij}$ is given as ϵ_{ij}/k or B_{ij} by using the coordinate transformation, the resulting coordinate shows the longitudinal noncommutativity in the form of (1.3) with $\theta^{ij} = -k\epsilon^{ij}$ or $\theta^{ij} = (B^{-1})^{ij}$, respectively. In the following we generalize the argument in [22]. Namely, by making the general coordinate transformation on the D-string world volume considered in the

previous section, we change the parametrization of the world volume from that in the static gauge described by (2.4) to that in a gauge in which $\mathcal{F}_{12}$ is equal to a certain constant value and reconsider the world volume theory in this gauge. Then it is found that if it is possible to fix the value of $\mathcal{F}_{12}$ arbitrarily not restricted to $1/k$ or B_{12}, we get a description with an arbitrary longitudinal noncommutativity of the same world volume theory.

Let us denote the D-string world volume coordinate in the static gauge considered in section 2 as x^i:

$$X^i(x) = x^i, \tag{3.17}$$

where X^i is the embedding function into the target space. In the static gauge a dynamical field on the world volume is a $U(1)$ gauge field $a_i(x)$. Combining the result in (2.4), the total field strength is given by

$$\mathcal{F}_{ij}(x) = \mathcal{F}'_{ij} + f_{ij}(x) = \mathcal{F}'_{ij} + \partial_{x^i} a_j(x) - \partial_{x^j} a_i(x), \tag{3.18}$$

where $\mathcal{F}'$ is a constant field strength defined in (2.10).

To begin with, we consider the case in which $a_i(x) = 0$. For an arbitrary non-zero constant antisymmetric tensor field $Y_{ij} = Y\epsilon_{ij}$ on the D-string world volume, we make a coordinate transformation on the world volume and choose its new coordinate σ^i under which a new field strength $\tilde{\mathcal{F}}_{ij}(\sigma)$ is given by Y_{ij}:

$$Y_{ij} = \tilde{\mathcal{F}}_{ij}(\sigma) = J_i^k J_j^l \mathcal{F}'_{kl}(x(\sigma)), \tag{3.19}$$

where J_i^k is the Jacobian matrix associated with our coordinate transformation:

$$J_i^k = \frac{\partial x^k}{\partial \sigma^i}. \tag{3.20}$$

Eq. (3.19) means that

$$Y = J\mathcal{F}'J^T, \tag{3.21}$$

and this is the condition which the coordinate σ, in other words, the coordinate transformation $x \to \sigma$, should satisfy. Below we will assume that for an arbitrary Y_{ij} we can choose such a coordinate σ^i. The special case $B = 0$ and $Y = 1/k$ is considered in [17, 18] and the case $k \to \infty$ and $Y = B_{12}$ corresponds to [22].

Next when there is a small fluctuation of $U(1)$ gauge field $a_i(x)$ in the static gauge (3.17), what is the counterpart of its degrees of freedom in the σ coordinate ($\tilde{\mathcal{F}} = Y$ gauge)? In the presence of the $a_i(x)$, we denote a new coordinate σ'^i which satisfies the gauge condition

$$Y_{ij} = \tilde{\mathcal{F}}_{ij}(\sigma') = J_i^k J_j^l \mathcal{F}_{kl}(x(\sigma')), \tag{3.22}$$

as

$$\sigma'^i = \sigma^i - d^i(\sigma). \tag{3.23}$$

Here σ^i is the coordinate which satisfies (3.19). $d^i(\sigma)$ defined here is a dynamical field corresponding to $a_i(x)$ in this gauge. Define

$$J'^k_i = \frac{\partial x^k}{\partial \sigma'^i} = J_i^k + \delta J_i^k, \tag{3.24}$$

then (3.22) can be rewritten as

$$\begin{aligned} Y_{ij} &= (J+\delta J)_i^k (J+\delta J)_j^l (\mathcal{F}'_{kl} + f_{kl}) \\ &= Y_{ij} + \delta J_i^k J_j^l \mathcal{F}'_{kl} + J_i^k \delta J_j^l \mathcal{F}'_{kl} + J_i^k J_j^l f_{kl}, \end{aligned} \tag{3.25}$$

here we have assumed that $\delta J \sim O(a)$ and kept terms up to the first order in a. This equation leads to

$$JfJ^T = -\delta J \mathcal{F}' J^T - J\mathcal{F}'\delta J^T. \tag{3.26}$$

We note that

$$J'^j_i = J_i^k + (\partial_i d^k(\sigma)) J_k^j, \quad \partial_i \equiv \frac{\partial}{\partial \sigma^i}, \tag{3.27}$$

and thus if we define $M_i^k = \partial_i d^k(\sigma)$, then $J' = (1+M)J$, namely, $\delta J = MJ$. Substituting this into (3.26) yields

$$JfJ^T = -MY + (MY)^T, \tag{3.28}$$

here we have used (3.21). For components, this equation gives

$$f_{12} = -Y_{12} \det J^{-1} \mathrm{tr} M = -Y_{12} \det J^{-1} \partial_i d^i. \tag{3.29}$$

Using $\det J = Y_{12}/\mathcal{F}'_{12}$ which can be easily obtained by (3.22), we find

$$-\mathcal{F}'_{12} \partial_i d^i = f_{12} = \partial_{x^1} a_2 - \partial_{x^2} a_1. \tag{3.30}$$

Noting that $\partial_{x^i} = (J^{-1})_i^j \partial_j$, this equation gives

$$Y_{12} \partial_i d^i = -\epsilon_{ij} \partial_i (J_j^k a_k). \tag{3.31}$$

Thus we get

$$d^i = (Y^{-1})^{ij} \partial_j x^k a_k = (Y^{-1})^{ij} \tilde{a}_j, \tag{3.32}$$

where we have introduced

$$\tilde{a}_j(\sigma) = \frac{\partial x^k}{\partial \sigma^j} a_k(x(\sigma)), \tag{3.33}$$

and ignored a difference of the $U(1)$ gauge degrees of freedom

$$d^i \to d^i + (Y^{-1})^{ij} \partial_j \lambda. \tag{3.34}$$

This result certainly reproduces those obtained in [17, 18, 22] as special cases and is a natural extension of theirs.

Now let us derive the action and its symmetry in the gauge (3.19). Following the argument in [22], we begin with the Born-Infeld action for our D-string in this gauge

$$S = T \int_\Sigma d^2\sigma \sqrt{\det(\tilde{g}_{ij} + 2\pi\alpha' \tilde{\mathcal{F}}_{ij})}, \tag{3.35}$$

where $\tilde{g}_{ij}$ is the induced metric for the embedding function $X^i(\sigma) = x^i(\sigma)$

$$\tilde{g}_{ij} = \partial_i x^a \partial_j x^b g_{ab}. \tag{3.36}$$

Using the gauge condition (3.19), this action becomes

$$
\begin{aligned}
S &= \text{const.} + \frac{T}{4}\sqrt{(2\pi\alpha')^2 Y_{12}^2}\int_\Sigma d^2\sigma \frac{1}{(2\pi\alpha')^2}(Y^{-1})^{il}(Y^{-1})^{jk}\partial_i x^a \partial_j x^b \partial_k x^c \partial_l x^d g_{ab} g_{cd} + \cdots \\
&= \text{const.} - \frac{T}{4}\sqrt{(2\pi\alpha')^2 Y_{12}^2}\int_\Sigma d^2\sigma \frac{1}{(2\pi\alpha')^2}\{x^a, x^d\}_{Y^{-1}}\{x^b, x^c\}_{Y^{-1}} g_{ab} g_{cd} + \cdots, \qquad (3.37)
\end{aligned}
$$

where we have introduced the Poisson bracket associated with Y^{-1} as

$$
\{A, B\}_{Y^{-1}} = i(Y^{-1})^{ij}\partial_i A \partial_j B. \qquad (3.38)
$$

In particular,

$$
\{\sigma^i, \sigma^j\}_{Y^{-1}} = i(Y^{-1})^{ij}. \qquad (3.39)
$$

In the presence of the small fluctuation $d^i(\sigma)$ defined in (3.23), we can see from (3.32) that the embedding coordinate becomes

$$
x^a \to x^a + \delta x^a = x^a + J_i^a (Y^{-1})^{ij}\tilde{a}_j. \qquad (3.40)
$$

Then it is easy to check that

$$
-i\{x^a, x^d\}_{Y^{-1}} \to (J^T Y^{-1} J)^{ad} - (J^T Y^{-1})^{ai}\tilde{f}_{ij}(Y^{-1}J)^{jd}, \qquad (3.41)
$$

where

$$
\tilde{f}_{ij} = \partial_i \tilde{a}_j - \partial_j \tilde{a}_i - i\{\tilde{a}_i, \tilde{a}_j\}_{Y^{-1}}. \qquad (3.42)
$$

Thus we obtain an action for the fluctuation $\tilde{a}_i$ as

$$
\frac{T}{4}\sqrt{\frac{Y_{12}^2}{(2\pi\alpha')^2}}\int_\Sigma d^2\sigma (J^T Y^{-1})^{aj}(\tilde{f}_{jl} - Y_{jl})(Y^{-1}J)^{ld}(J^T Y^{-1})^{bk}(\tilde{f}_{ki} - Y_{ki})(Y^{-1}J)^{ic} g_{ab} g_{cd}. \qquad (3.43)
$$

Define

$$
\begin{aligned}
G^{ij} &= -\frac{1}{(2\pi\alpha')^2}(Y^{-1}J)^{ia} g_{ab} (J^T Y^{-1})^{bj} = -\frac{1}{(2\pi\alpha')^2}(Y^{-1}\tilde{g}Y^{-1})^{ij}, \qquad (3.44) \\
G_s &= g_s \det(2\pi\alpha' Y \tilde{g}^{-1})^{\frac{1}{2}}, \qquad (3.45)
\end{aligned}
$$

then the action takes the form of

$$
\begin{aligned}
&\frac{1}{g_{YM}^2}\int_\Sigma d^2\sigma \sqrt{\det G_{ij}} G^{ij} G^{kl} \frac{1}{4}(\tilde{f}_{ik} - Y_{ik})(\tilde{f}_{jl} - Y_{jl}) \\
&\quad = \frac{1}{g_{YM}^2}\int_\Sigma d^2\sigma \sqrt{\det G_{ij}} G^{ij} G^{kl} \frac{1}{4}\tilde{f}_{ik}\tilde{f}_{jl} + \text{total derivative}, \qquad (3.46)
\end{aligned}
$$

where

$$
g_{YM}^2 = \frac{2\pi\alpha'}{G_s}. \qquad (3.47)
$$

This is the action of the NCYM theory with $\theta = Y^{-1}$ up to the second order in Y^{-1}. We believe that generalization of the arguments given in [25] would work in our case and it

provides the exact NCYM theory with $\theta = Y^{-1}$ in which the Poisson bracket (3.38) is replaced by the Moyal bracket. Thus we have verified that at least for small fluctuations the D-string world volume theory in the gauge (3.22) is described by the NCYM theory with noncommutativity Y^{-1} and that $\tilde{a}_i$ corresponds to the NC gauge field. It is worth noticing that (3.44) and (3.45) are natural generalization of the open string metric and open string coupling constant in the zero-slope limit given in [11].

As another evidence that we have NCYM theory with $\theta = Y^{-1}$, let us analyze the symmetry of (3.46). It is argued in [17, 18, 22] that the gauge condition of the form (3.19) does not completely fix the whole reparametrization invariance and there is a residual one, which can be interpreted as the noncommutative gauge invariance (1.2) in terms of $\tilde{a}_i$ defined like (3.40). In the present case, we also have residual diffeomorphism invariance after fixing the gauge as in (3.19). Namely, if we make the coordinate transformation $\sigma^i \to \sigma^i + V^i(\sigma)$ using a vector field $V^i(\sigma)$ on the world volume, the condition that this coordinate preserves the gauge condition is

$$Y_{kj}\partial_i V^k(\sigma) + Y_{ik}\partial_j V^k(\sigma) = 0, \tag{3.48}$$

which implies that there exists a scalar field $\rho(\sigma)$ on the world volume such that

$$V^i = (Y^{-1})^{ij}\partial_j \rho. \tag{3.49}$$

Then $x^i(\sigma)$ is transformed as

$$x^i \to x^i - (Y^{-1})^{jk}\partial_j\rho\partial_k x^i = x^i + i\{\rho, x^i\}_{Y^{-1}}. \tag{3.50}$$

This and (3.39) strongly suggest that the world volume coordinate in the gauge (3.19) has the longitudinal noncommutativity $\theta = Y^{-1}$ in the form of (1.3). If there exists the fluctuation $\tilde{a}_i(\sigma)$, x^i is modified according to (3.40) and the transformation (3.50) becomes

$$x'^i \to x'^i + i\{\rho, x'^i\}_{Y^{-1}} \equiv x'^i + \delta x'^i. \tag{3.51}$$

In terms of $\tilde{a}_i(\sigma)$, (3.51) can be rewritten as

$$\delta\tilde{a}_i = \partial_i\rho + i\{\rho, \tilde{a}_i\}_{Y^{-1}}, \tag{3.52}$$

where we have used $\delta x'^i = J^i_j (Y^{-1})^{jk}\delta\tilde{a}_k$ and $\partial_i\partial_j x^k = 0$, because it is possible to choose J as a constant matrix in the case of a constant $\mathcal{F}'$.

Thus we conclude that for an arbitrary non-zero constant antisymmetric field Y_{ij}, if we choose a coordinate σ such that $\tilde{\mathcal{F}}(\sigma) = Y$ by means of the general coordinate transformation on the world volume, the world volume theory in this coordinate is described by the NCYM theory with $\theta = Y^{-1}$ and correspondingly, the world volume has the longitudinal noncommutativity $(Y^{-1})^{ij}$.

We have seen that the longitudinal noncommutativity θ can be arbitrarily varied by the way of fixing the diffeomorphism invariance on the world volume. As emphasized in the previous section, from the point of view of NCYM theory, the magnetic background Φ is responsible for the degrees of freedom to change the value of θ. Therefore, it is conjectured that the degrees of freedom of Φ should correspond to those of diffeomorphism on the world

volume which is noncommutative in general. In order to see that this is the case, let us briefly discuss the derivation of (2.16). The following argument is a generalization of that given in [21]. It is natural to guess that the D-string configuration in the $\tilde{F}' = Y$ gauge is equivalent to that in the static gauge in the background metric $g_{ij} = G'_{ij}$, antisymmetric tensor field $\tilde{F}' - Y$, and $U(1)$ field strength Y. The T-duality transformation in 1,2-directions maps this to a noncommutative D-instanton configuration

$$[\bar{q}^1, \bar{q}^2] = -i(Y^{-1})^{12}, \tag{3.53}$$

in the T-dual background

$$\bar{G}'^{ij} + 2\pi\alpha' \bar{B}^{ij} = \frac{1}{G' + 2\pi\alpha'(\tilde{F}' - Y)}. \tag{3.54}$$

Similarly to above and [21], for fluctuations $\delta\bar{q}^i$ around (3.53), we define $\bar{a}_i$ as

$$\delta\bar{q}^i = (Y^{-1})^{ij}\bar{a}_j(\bar{q}). \tag{3.55}$$

Substituting this in the original effective action of D-instantons for slowly varying fields[‖]

$$S \sim \mathrm{Tr}\sqrt{\det(\bar{G}^{ij} + 2\pi\alpha'\bar{B}^{ij} + i(2\pi\alpha')^{-1}[\bar{q}^i, \bar{q}^j])}, \tag{3.56}$$

and expressing the functions of $\bar{q}^i$ as c-number functions whose product is given by the $*$-product in (1.1) with $\theta = Y^{-1}$, we get the noncommutative Born-Infled action in the following form:

$$S_\Phi \sim \sqrt{\det(\hat{G} + 2\pi\alpha'(f + \Phi))}, \tag{3.57}$$

where

$$\Phi = -Y - (2\pi\alpha')^2 Y\bar{B}Y, \tag{3.58}$$

which reproduces (2.16) with $\theta = Y^{-1}$.

Now let us summarize above arguments using boundary states in accordance with [17, 18]. As constructed in [17], a boundary state corresponding to (2.1) is given by

$$|B\rangle_{-1} = \mathrm{tr}P\exp\left(-i\int_0^{2\pi} dsP_i(s)q^i\right)|X^i = 0\rangle_{-1}, \tag{3.59}$$

where P_i is the canonical momentum of string and $|X^i = 0\rangle_{-1}$ is the Dirichlet boundary state:

$$X^i(s)|X^i = 0\rangle_{-1} = 0. \tag{3.60}$$

In the path integral representation, (3.59) can be rewritten as

$$|B\rangle_{-1} = \int [dq^1 dq^2]\exp\left(\frac{i}{k}\int_0^{2\pi} dsq^1(s)\partial_s q^2(s) - i\int_0^{2\pi} dsP_i(s)q^i(s)\right)|X^i = 0\rangle_{-1}. \tag{3.61}$$

Generalizing this, a general configuration of D-instantons

$$X^\mu = \phi^\mu(q) \quad (\mu \geq 1), \tag{3.62}$$

[‖]We adopt a naive generalization of the abelian Born-Infeld action.

should correspond to a boundary state

$$|B\rangle_{-1} = \int [dq^1 dq^2] \exp\left(\frac{i}{k}\int_0^{2\pi} dsq^1(s)\partial_s q^2(s) - i\int_0^{2\pi} dsP_\mu(s)\phi^\mu(q^i(s))\right)|X^i = 0\rangle_{-1}. \tag{3.63}$$

On the other hand, in the D-string picture, the world volume theory consists of a $U(1)$ gauge field A_i on the world volume and scalar fields ϕ^a ($a \geq 3$) which describe the transverse coordinates of the world volume. In the static gauge $\phi^i(q) = q^i$, the boundary state corresponding to this general configuration is given by

$$\begin{aligned} |B\rangle_1 &= \int [dq^1 dq^2] \exp\left(i\int_0^{2\pi} dsA_i(q(s))\partial_s q^i(s) - i\int_0^{2\pi} ds(P_i q^i(s) + P_a(s)\phi^a(q(s)))\right) \\ &\quad \times |X^i = 0\rangle_{-1}, \end{aligned} \tag{3.64}$$

which coincides with (3.63) under an identification $F_{12} = 1/k$. Thus we see that in both boundary state q^i plays a role of the parametrization of the world volume. Now following [18], let us consider the most general boundary state which includes all of the fields A_i, ϕ^μ which appeared in (3.63), (3.64)

$$|B\rangle = \int [dq^1 dq^2] \exp\left(i\int_0^{2\pi} dsA_i(q(s))\partial_s q^i(s) - i\int_0^{2\pi} dsP_\mu(s)\phi^\mu(q(s))\right)|X^i = 0\rangle_{-1}. \tag{3.65}$$

Then it is easy to check that this boundary state is invariant under the reparametrization of the world volume coordinate q^i[18]. Moreover, by using this invariance, when we fix q^i in the static gauge $\phi^i(q) = q^i$, (3.65) reproduces (3.64), while if we fix q^i in such a way that $F_{12} = 1/k$, it reproduces (3.63). In the latter case, as we have seen in (3.39), q^i become noncommutative coordinates. In this sense, an original configuration described by (3.65) has the invariance of the diffeomorphism on the (noncommutative, in general) world volume [18]. According to the notation in [17], we denote the group which consists of such diffeomorphisms as $Diff$. $Diff$ has a subgroup of diffeomorphisms which preserve the value of $\mathcal{F}$. We denote it as $Diff_{\mathcal{F}}$. This is exactly the symmetry in the D-instanton picture and is inherited from the original $U(\infty)$ symmetry[18]

$$\delta q^i = i[\epsilon, q^i]. \tag{3.66}$$

As we showed above, $Diff_{\mathcal{F}}$ can be also interpreted as the noncommutative gauge symmetry[17, 18, 22]. Arguments in this section implies that the residual symmetry of the reparametrization invariance which the original system (3.65) has, namely,

$$\frac{Diff}{Diff_{\mathcal{F}}}, \tag{3.67}$$

corresponds to a hidden 'symmetry' which is responsible for changing the longitudinal noncommutativity and is conjectured to be equivalent to the degrees of freedom of Φ.

4 Discussions

Let us discuss the meaning of our results from the point of view of searching a nonperturbative mechanism and definition of string theory.

We have shown that the 'symmetry' to change the longitudinal noncommutativity results from the degrees of freedom of the way of choosing parametrizations on a noncommutative world volume. Thus starting with a configuration like (3.65) gives a clue to the proof of (2.8) or (2.9). In fact, there have been already some results in this approach[25]. Moreover, it is expected to clarify relations or symmetries between various noncommutativities. In particular, although we concentrated on the relation between the transverse noncommutativity and the longitudinal one in the above discussion using boundary states, it would be interesting to construct a boundary state which also includes the momentum noncommutativity.** If it is possible, it would make clear interrelations between the noncommutativities and serve for the nonperturbative formulation of string theory.

On the other hand, from the point of view of D-brane matrix models, the noncommutativities may provide natural regularizations. If we see the lower-dimensional D-brane matrix models as candidates for the nonperturbative definition of string theory like[1, 2, 3], these must be well-defined field theories. It would be interesting to examine a possibility that various noncommutativities associated with D-branes provide natural regularizations for them. In this sense, it would be important to define a field theory on a noncommutative space and to clarify a role played by the noncommutativity as a regularization [26].

In relation to the work of [11], we believe that the gauge choice (3.19) would correspond to a certain regularization of the two-dimensional field theory which interpolates between the Pauli-Villars and the point-splitting, for example, a hybrid point splitting proposed in [27]. We hope to return to this issue in future works.

Acknowledgements

The author would like to thank O. Andreev, M. Kato and N. Ishibashi for valuable discussions and useful comments. This work was also influenced so much by the Summer Institute '99 at Yamanashi and the workshop at YITP, Japan. The author is grateful to the organizers of these workshops for providing such opportunities.

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** For example, in [5] such a boundary state is concretely constructed.

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